ON LOCAL DEFORMATION PROPERTY FOR UNIFORM EMBEDDINGS

TATSUHIKO YAGASAKI

ABSTRACT. The local deformation property for uniform embeddings is studied in a formal view point and its behaviour under the restriction and union of domains of embeddings is clarified. As a typical example, any manifold with a geometric group action is shown to have this property based on the additivity of this deformation property and the free case which is shown in a previous paper in term of metric covering spaces over compact manifolds

1. INTRODUCTION

This article is a continuation of study of topological properties of spaces of uniform embeddings and groups of uniform homeomorphisms ([2, 5, 8]). Since the notion of uniform continuity depends on the choice of metrics, it is essential to select reasonable classes of metric spaces (X, d). In [2] (cf, [6, Section 5.6]) A.V. Černavskiĭ considered the case where X is the interior of a compact manifold N and the metric d is a restriction of some metric on N. On the other hand, in the previous paper [8] we considered metric covering spaces over compact manifolds and obtained a local deformation theorem for uniform embeddings in those spaces.

In this article we study the local deformation property for uniform embeddings in a formal view point and clarify its behaviour under the restriction and union of domains of embeddings. This enables us to deduce this deformation property for more complicated metric spaces to which the typical deformations theorem mentioned in the last paragraph can not be applied directly. As a typical example we consider the class of manifolds with locally geometric group actions (Thereom 4.1). Here a group action on a locally compact metric space is called (locally) geometric if it is proper, cocompact and (locally) isometric. In term of covering transformation groups, metric covering spaces over compact manifolds corresponds to manifolds with free locally geometric group actions. Therefore, this is regarded as a generalization from the free case to the non-free case. The key observation is that any metric space with a geometric group action is a finite union of invariant open subsets each of which is a trivial metric covering space. Thus the non-free case follows from the free case and the additivity of the deformation property. We also clarify the relation between the deformation property of a manifold and that of neighborhoods of its ends.

This paper is organized as follows. Section 2 includes basic definitions on uniform embeddings and the results on metric covering spaces obtained in [8]. In Section 3 we study the behaviour of local deformation property for uniform embeddings under the restriction and union. In the final Section

4 we discuss examples of manifolds with this deformation property, especially manifolds with locally geometric group actions.

2. Spaces of uniform embeddings

In this section we recall basic definitions on uniform embeddings and the results on metric covering spaces obtained in [8].

2.1. Conventions.

In this article, maps between topological spaces are always assumed to be continuous. For a topological space X and a subset A of X, the symbols $\mathrm{Int}_X A$, $cl_X A$ and $\mathrm{Fr}_X A$ denote the topological interior, closure and frontier of A in X.

Suppose X is a metric space. The metric of X is denoted by the symbol d_X, otherwise specified. For $x \in X$ and $\varepsilon \in [0, \infty]$ let $O_\varepsilon(x)$ denote the open ε-ball in X centered at the point x. (Note that $O_0(x) = \emptyset$ and $O_\infty(x) = X$.) As usual, the metric of any subset A of X is given by the restriction of the metric d_X to A. The distance between two subsets A and B of X is defined by $d(A, B) = \inf\{d_X(a, b) \mid a \in A, b \in B\}$. The open ε-neighborhood of A in X is defined by

$$O_\varepsilon(A) = O_\varepsilon(A, X) = \{x \in X \mid d(x, A) < \varepsilon\}.$$

Note that $d(x, A) < \varepsilon$ if and only if $d_X(x, a) < \varepsilon$ for some $a \in A$ and that $O_\varepsilon(O_\delta(A)) \subset O_{\varepsilon+\delta}(A)$ and $O_\varepsilon(A, Y) = O_\varepsilon(A) \cap Y$ if $A \subset Y \subset X$. We write $A \subset_u B$ in X and call B a uniform neighborhood of A in X if $O_\varepsilon(A) \subset B$ for some $\varepsilon > 0$. Note that if $A \subset_u B$ in X, then $A \cap Y \subset_u B \cap Y$ in Y for any $Y \subset X$. In fact, if $O_\varepsilon(A) \subset B$, then $O_\varepsilon(A \cap Y, Y) \subset O_\varepsilon(A) \cap Y \subset B \cap Y$.

We say that a subset A of X is ε-discrete if $d_X(x, y) \geq \varepsilon$ for any distinct points $x, y \in A$ and that A is uniformly discrete if it is ε-discrete for some $\varepsilon > 0$. More generally a family $\{F_\lambda\}_{\lambda \in \Lambda}$ of subsets of X is said to be ε-discrete if $d(F_\lambda, F_\mu) \geq \varepsilon$ for any distinct $\lambda, \mu \in \Lambda$.

2.2. Spaces of uniform embeddings.

Below the symbols X, Y, Z denote any metirc spaces. A map $f : X \to Y$ is said to be uniformly continuous if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $x, x' \in X$ and $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \varepsilon$. The map f is called a uniform homeomorphism if f is bijective and both f and f^{-1} are uniformly continuous. A uniform embedding is a uniform homeomorphism onto its image.

Let $\mathcal{C}^u(X, Y)$ denote the space of uniformly continuous maps $f : X \to Y$. The metric d_Y on Y induces the sup-metric d on $\mathcal{C}^u(X, Y)$ defined by

$$d(f, g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\} \in [0, \infty].$$

The topology on $\mathcal{C}^u(X, Y)$ induced by this sup-metric d is called the uniform topology. Below the space $\mathcal{C}^u(X, Y)$ and its subspaces are endowed with the sup-metric d and the uniform topology, otherwise specified. The composition map

$$\mathcal{C}^u(X, Y) \times \mathcal{C}^u(Y, Z) \longrightarrow \mathcal{C}^u(X, Z) : (f, g) \longmapsto gf$$

is continuous.

Let $\mathcal{H}^u(X)$ denote the group of uniform homeomorphisms h of X onto itself (endowed with the sup-metric and the uniform topology). The group $\mathcal{H}^u(X)$ is a topological group.

For $A, B \subset X$, let $\mathcal{E}^u(A, X; B)$ denote the space of uniform embeddings $f : A \to X$ with $f = \mathrm{id}$ on $A \cap B$ (with the sup-metric and the uniform topology). When X is a topological n-manifold possibly with boundary, an embedding $f : A \to X$ is said to be quasi-proper if $f(A \cap \partial X) \subset \partial X$. Let $\mathcal{E}^u_{\#}(A, X; B)$ denote the subspace of $\mathcal{E}^u(A, X; B)$ consisting of quasi-proper uniform embeddings and for $\varepsilon > 0$ let $\mathcal{E}^u_{\#}(i_A, \varepsilon; A, X; B)$ denote the open ε-neighborhood of the inclusion map $i_A : A \subset X$ in the space $\mathcal{E}^u_{\#}(A, X; B)$.

2.3. **Uniform embeddings in metric covering spaces.**

In [8] we introduced the notion of metric covering projections as the C^0-version of Riemannian coverings in the smooth category. For the basics on covering spaces, we refer to [7, Chapter 2, Section 1]. Note that if $p : M \to N$ is a covering projection and N is a topological n-manifold possibly with boundary, then so is M and $\partial M = \pi^{-1}(\partial N)$.

Definition 2.1. A map $\pi : X \to Y$ between metric spaces is called a metric covering projection if it satisfies the following conditions:

$(*)_1$ There exists an open cover $\mathcal{U}$ of Y such that for each $U \in \mathcal{U}$ the inverse $\pi^{-1}(U)$ is the disjoint union of open subsets of X each of which is mapped isometrically onto U by π.

$(*)_2$ For each $y \in Y$ the fiber $\pi^{-1}(y)$ is uniformly discrete in X.

$(*)_3$ $d_Y(\pi(x), \pi(x')) \leq d_X(x, x')$ for any $x, x' \in X$.

In [4, Theorem 5.1] R. D. Edwards and R. C. Kirby obtained a fundamental local deformation theorem for embeddings of a compact subspace in a manifold. In [8, Theorem 1.1] from this deformation theorem we deduced a local deformation theorem for uniform embeddings in any metric covering space over a compact manifold. Here, in order to pass from the compact case to the uniform case, the Arzela-Ascoli theorem (cf. [3, Theorem 6.4]) played an essential role.

Theorem 2.1. *Suppose $\pi : (M, d) \to (N, \rho)$ is a metric covering projection, N is a compact n-manifold possibly with boundary, X is a subset of M, $W' \subset W$ are uniform neighborhoods of X in (M, d) and Z, Y are subsets of M such that Y is a uniform neighborhood of Z. Then there exists a neighborhood $\mathcal{W}$ of the inclusion map $i_W : W \subset M$ in $\mathcal{E}^u_{\#}(W, M; Y)$ and a homotopy $\varphi : \mathcal{W} \times [0, 1] \longrightarrow \mathcal{E}^u_{\#}(W, M; Z)$ such that*

(1) *for each $h \in \mathcal{W}$* (i) $\varphi_0(h) = h$, (ii) $\varphi_1(h) = \mathrm{id}$ *on* X,

(iii) $\varphi_t(h) = h$ *on* $W - W'$ *and* $\varphi_t(h)(W) = h(W)$ $(t \in [0, 1])$,

(iv) *if $h = \mathrm{id}$ on $W \cap \partial M$, then $\varphi_t(h) = \mathrm{id}$ on $W \cap \partial M$ $(t \in [0, 1])$,*

(2) $\varphi_t(i_W) = i_W$ $(t \in [0, 1])$.

In [2] it is shown that $\mathcal{H}^u(M, d)$ is locally contractible in the case where M is the interior of a compact manifold N and the metric d is a restriction of some metric on N. As a direct consequence

of Theorem 2.1 it follows that $\mathcal{H}^u(M, d)$ is locally contractible if (M, d) admits a metric covering projection $\pi : (M, d) \to (N, \rho)$ onto a compact n-manifold N possibly with boundary.

3. Local deformation property for uniform embeddings

In this section we study the formal behaviour of local deformation property for uniform embeddings. Throughout this section (M, d) denotes a topological n-manifold possibly with boundary with a fixed metric d.

3.1. **Basic definitions.**

For the sake of notational simplicity and to clarify the essence of the arguments in our formal approach, we use the following notations, which are motivated by Theorem 2.1.

Definition 3.1. (Admissible deformations)

(1) A tuple (X, W', W, Z, Y, M) is said to be admissible if $X \subset_u W' \subset W \subset M$ and $Z \subset_u Y \subset M$.

(2) Suppose (X, W', W, Z, Y, M) is an admissible tuple and $\mathcal{W}$ is a neighborhood of the inclusion map $i_W : W \subset M$ in $\mathcal{E}^u_\#(W, M; Y)$. A homotopy

$$\varphi : \mathcal{W} \times [0, 1] \longrightarrow \mathcal{E}^u_\#(W, M; Z)$$

is called an admissible deformation of $\mathcal{W}$ on X if it satisfies the following conditions:

$(*_1)$ for each $h \in \mathcal{W}$ (i) $\varphi_0(h) = h,$ (ii) $\varphi_1(h) = \mathrm{id}$ on $X,$

(iii) $\varphi_t(h) = h$ on $W - W'$ and $\varphi_t(h)(W) = h(W)$ $(t \in [0, 1]),$

(iv)   if $h = \mathrm{id}$ on $W \cap \partial M$, then $\varphi_t(h) = \mathrm{id}$ on $W \cap \partial M$ $(t \in [0, 1]),$

$(*_2)$ $\varphi_t(i_W) = i_W$ $(t \in [0, 1]).$

We use the symbol LD as the abbreviation of the phrase "Local deformation property for uniform embedding".

Definition 3.2. (Local deformation property for uniform embeddings)

(1) For an admissible tuple (X, W', W, Z, Y, M) the condition $\mathrm{LD}(X, W', W, Z, Y, M)$ is defined by $\mathrm{LD}(X, W', W, Z, Y, M) \iff$ There exists a neighborhood $\mathcal{W}$ of i_W in $\mathcal{E}^u_\#(W, M; Y)$ and an admissible deformation φ of $\mathcal{W}$ on X.

(2) For $X \subset M$ the condition $\mathrm{LD}(X, M)$ is defined by

   $\mathrm{LD}(X, M) \iff \mathrm{LD}(X, W', W, Z, Y, M)$ holds for any admissible tuple (X, W', W, Z, Y, M).

(3) For $A \subset M$ the condition $A : (\mathrm{LD})_M$ is defined by

   $A : (\mathrm{LD})_M \iff \mathrm{LD}(X, M)$ holds for any $X \subset A$

     $(\iff$ for any admissible tuple (X, W', W, Z, Y, M) with $X \subset A$

       there exists a neighborhood $\mathcal{W}$ of i_W in $\mathcal{E}^u_\#(W, M; Y)$ and

       an admissible deformation $\varphi : \mathcal{W} \times [0, 1] \longrightarrow \mathcal{E}^u_\#(W, M; Z)$ of $\mathcal{W}$ on $X.)$

(4) We write $M : (\mathrm{LD})$ and say that M has the local deformation property for uniform embeddings if $M : (\mathrm{LD})_M$.

Remark 3.1. If $\mathrm{LD}(M, M, M, \emptyset, \emptyset, M)$ holds, then $\mathcal{H}^u(M)$ is locally contractible. In fact, since $\mathcal{H}^u(M) \subset \mathcal{E}^u_\#(M, M)$, any admissible deformation φ of a neighborhood $\mathcal{W}$ of id_M in $\mathcal{E}^u_\#(M, M)$ on M restricts to a contraction of the neighborhood $\mathcal{W} \cap \mathcal{H}^u(M)$ of id_M in $\mathcal{H}^u(M)$ rel id_M.

Remark 3.2. Suppose N is another topological n-manifold possibly with boundary with a fixed metric and $h : M \to N$ is a uniform homeomorphism. Then $M : (\mathrm{LD}) \iff N : (\mathrm{LD})$. In fact, if (X, W', W, Z, Y, M) is an admissible tuple in M, then its image $(X_1, W'_1, W_1, Z_1, Y_1, N)$ under h is an admissible tuple in N and h induces a homeomorphism

$$\chi : \mathcal{E}^u_\#(W, M) \cong \mathcal{E}^u_\#(W_1, N) : \quad \chi(f) = hfh^{-1}.$$

Moreover, if $\varphi_t : \mathcal{W} \to \mathcal{E}^u_\#(W, M, Z)$ is an admissible deformation of a neighborhood $\mathcal{W}$ of i_W in $\mathcal{E}^u_\#(W, M, Y)$ on X, then $\mathcal{W}_1 = \chi(\mathcal{W})$ is a neighborhood of i_{W_1} in $\mathcal{E}^u_\#(W_1, N, Y_1)$ and an admissible deformation $\psi_t : \mathcal{W}_1 \to \mathcal{E}^u_\#(W_1, N, Z_1)$ of $\mathcal{W}_1$ on X_1 is defined by $\psi_t = \chi \varphi_t \chi^{-1}$.

3.2. Basic lemmas.

Lemma 3.1. *Suppose (X, W', W, Z, Y, M) is an admissible tuple.*

(1) *Suppose the condition $\mathrm{LD}(X, W', W, Z, Y, M)$ holds.*
  (i) *If $W' \subset W'_1 \subset W \subset M$, then $\mathrm{LD}(X, W'_1, W, Z, Y, M)$ holds.*
  (ii) *If $W' \subset_u W \subset W_1 \subset M$, then $\mathrm{LD}(X, W', W_1, Z, Y, M)$ holds.*
  (iii) *If $Z_1 \subset Z$ and $Y \subset Y_1 \subset M$, then $\mathrm{LD}(X, W', W, Z_1, Y_1, M)$ holds.*

(2) *Suppose $W \subset_u N \subset M$ and N itself is a topological n-manifold possibly with boundary. Then*
  (i) (a) *$(X, W', W, Z \cap N, Y \cap N, N)$ is also an admissible tuple,* (b) *$W \cap \partial N = W \cap \partial M$,*
  (ii) *$\mathrm{LD}(X, W', W, Z \cap N, Y \cap N, N) \iff \mathrm{LD}(X, W', W, Z, Y, M)$.*

Proof. (1) The condition $\mathrm{LD}(X, W', W, Z, Y, M)$ yields a neighborhood $\mathcal{W} = \mathcal{E}^u_\#(i_W, \gamma, W, M; Y)$ of i_W in $\mathcal{E}^u_\#(W, M; Y)$ and an admissible deformation $\varphi : \mathcal{W} \times [0, 1] \longrightarrow \mathcal{E}^u_\#(W, M; Z)$. In each case of (i), (ii) and (iii) the required neighborhood $\mathcal{W}_1$ and an admissible deformation ψ of $\mathcal{W}_1$ on X are obtained as follows:

(i) $\mathcal{W}$ and φ themselves satisfy the required condition.

(ii) Consider the neighborhood $\mathcal{W}_1 := \mathcal{E}^u_\#(i_{W_1}, \gamma; W_1, M; Y)$ of i_{W_1} in $\mathcal{E}^u_\#(W_1, M; Y)$ and define a homotopy $\psi : \mathcal{W}_1 \times [0, 1] \longrightarrow \mathcal{E}^u_\#(W_1, M; Z)$ by

$$\psi_t(h) = \begin{cases} \varphi_t(h|_W) & \text{on } W \\ h & \text{on } W_1 - W'. \end{cases}$$

It is a routine work to verify that ψ is well-defined and satisfies the admissibility condition. The uniform continuity of $\psi_t(h)$ follows from those of $\varphi_t(h|_W)$ and h and the condition $W' \subset_u W$. For the inverse map $\psi_t(h)^{-1} : h(W_1) \to W_1$, consider the factorization $\psi_t(h)^{-1} = (\psi_t(h)^{-1}h)h^{-1}$. Since

$$\psi_t(h)^{-1}h = \begin{cases} \varphi_t(h|_W)^{-1}h|_W & \text{on } W \\ \mathrm{id} & \text{on } W_1 - W', \end{cases}$$

the uniform continuity of $\psi_t(h)^{-1}h$ and $\psi_t(h)^{-1}$ follow from those of $\varphi_t(h|_W)^{-1}$, h and h^{-1}.

(iii) Since $\mathcal{E}_\#^u(W, M; Y_1) \subset \mathcal{E}_\#^u(W, M; Y)$ and $\mathcal{E}_\#^u(W, M; Z) \subset \mathcal{E}_\#^u(W, M; Z_1)$, it follows that $\mathcal{W}_1 = \mathcal{W} \cap \mathcal{E}_\#^u(W, M; Y_1)$ is a neighborhood of i_W in $\mathcal{E}_\#^u(W, M; Y_1)$ and the admissible deformation

$$\psi : \mathcal{W}_1 \times [0, 1] \longrightarrow \mathcal{E}_\#^u(W, M; Z_1)$$

of $\mathcal{W}_1$ on X is defined by $\psi_t(h) = \varphi_t(h)$ $(h \in \mathcal{W}_1)$.

(2)(i) (a) Note that $X \subset_u W'$ and $Z \cap N \subset_u Y \cap N$ in N.

(b) Since $W \subset_u N$ in M, it follows that $W \subset V \subset N$ for some open subset V of M. Then $\partial V = V \cap \partial N = V \cap \partial M$ and so $W \cap \partial N = W \cap \partial M$.

(ii) Since $W \cap (N \cap Y) = W \cap Y$ and $W \cap (N \cap Z) = W \cap Z$, the inclusion $i_N : N \subset M$ induces the isometric embeddings

$$i_{N*} : \mathcal{E}_\#^u(W, N; Y \cap N) \subset \mathcal{E}_\#^u(W, M; Y) \quad \text{and} \quad i_{N*} : \mathcal{E}_\#^u(W, N; Z \cap N) \subset \mathcal{E}_\#^u(W, M; Z).$$

Since $W \subset_u N$, it follows that $O_\varepsilon(W) \subset N$ for some $\varepsilon > 0$ and for any $\delta \in (0, \varepsilon)$ we have the canonical identification

$$i_{N*} : \mathcal{W}_\delta' \equiv \mathcal{E}_\#^u(i_W, \delta, W, N; Y \cap N) \cong \mathcal{W}_\delta \equiv \mathcal{E}_\#^u(i_W, \delta, W, M; Y).$$

If $\mathcal{W}_\delta'$ admits an admissible deformation $\psi : \mathcal{W}_\delta' \times [0, 1] \longrightarrow \mathcal{E}_\#^u(W, N; Z \cap N)$ on X, then an admissible deformation of $\mathcal{W}_\delta$ on X is defined by

$$\varphi : \mathcal{W}_\delta \times [0, 1] \longrightarrow \mathcal{E}_\#^u(W, M; Z) : \varphi_t(h) = \psi_t(h) \ (= i_N \psi_t((i_{N*})^{-1}h)).$$

Conversely, if $\mathcal{W}_\delta$ admits an admissible deformation $\varphi : \mathcal{W}_\delta \times [0, 1] \longrightarrow \mathcal{E}_\#^u(W, M; Z)$ on X, then an admissible deformation of $\mathcal{W}_\delta'$ on X is defined by

$$\psi : \mathcal{W}_\delta' \times [0, 1] \longrightarrow \mathcal{E}_\#^u(W, N; Z \cap N) : \psi_t(h) = \varphi_t(h) \ (= (i_{N*})^{-1} \varphi_t(i_N h)).$$

Note that $\varphi_t(h)(W) = h(W) \subset N$. This completes the proof. $\square$

Lemma 3.2. *Suppose $X \subset_u N \subset M$ and N itself is a topological n-manifold possibly with boundary. Then,* $\mathrm{LD}(X, N) \iff \mathrm{LD}(X, M)$.

Proof. Since $X \subset_u N$, we have $O_\varepsilon(X) \subset N$ for some $\varepsilon > 0$.

(1) $\mathrm{LD}(X, N) \implies \mathrm{LD}(X, M)$:

Take any admissible tuple (X, W', W, Z, Y, M) in M. There exists $\delta \in (0, \varepsilon/2)$ such that $O_{2\delta}(X) \subset W'$. Consider the admissible tuple $(X, O_\delta(X), O_{2\delta}(X), Z, Y, M)$ in M. Note that $O_{2\delta}(X) \subset_u N$ since $2\delta < \varepsilon$ and that $Z \cap N \subset_u Y \cap N$ in N. Hence, the condition $\mathrm{LD}(X, O_\delta(X), O_{2\delta}(X), Z \cap N, Y \cap N, N)$ holds by the assumption $\mathrm{LD}(X, N)$ and the condition $\mathrm{LD}(X, O_\delta(X), O_{2\delta}(X), Z, Y, M)$ also holds by Lemma 3.1 (2). Then Lemma 3.1 (1) (i), (ii) imply the condition $\mathrm{LD}(X, W', W, Z, Y, M)$ as required.

(2) $\mathrm{LD}(X, M) \implies \mathrm{LD}(X, N)$:

Take any admissible tuple (X, W', W, Z, Y, N) in N. There exists $\delta \in (0, \varepsilon/2)$ such that $O_{2\delta}(X, N) \subset W'$ and $O_\delta(Z, N) \subset Y$. Note that $O_{2\delta}(X, N) = O_{2\delta}(X) \subset_u N$ and $O_\delta(Z, N) = O_\delta(Z) \cap N$. Consider the admissible tuple $(X, O_\delta(X), O_{2\delta}(X), Z, O_\delta(Z), M)$ in M. Then, by the assumption

$\text{LD}(X, M)$ the condition $\text{LD}(X, O_\delta(X), O_{2\delta}(X), Z, O_\delta(Z), M)$ holds and by Lemma 3.1 (2) the condition $\text{LD}(X, O_\delta(X), O_{2\delta}(X), Z, O_\delta(Z, N), N)$ also holds. Hence Lemma 3.1 (1) implies the condition $\text{LD}(X, W', W, Z, Y, M)$ as required. $\square$

3.3. **Main propositions.**

Proposition 3.1. (1) (i) *Suppose* $A \subset B \subset M$. *Then* $B : (\text{LD})_M \implies A : (\text{LD})_M$.

(ii) *Suppose* $A \subset_u N \subset M$ *and* N *itself is a topological n-manifold possibly with boundary. Then,* $A : (\text{LD})_N \iff A : (\text{LD})_M$.

(2) (i) *Suppose* $A \subset_u U \subset M$ *and* $B \subset M$. *Then* $U, B : (\text{LD})_M \implies A \cup B : (\text{LD})_M$.

(ii) *If* $A_1, \cdots, A_m$ *are subsets of* M *and each* A_i *admits a uniform neighborhood which satisfies the condition* $(\text{LD})_M$, *then the union* $\cup_{i=1}^m A_i$ *satisfies the condition* $(\text{LD})_M$.

(3) *Suppose* K *is a relatively compact subset of* M. *Then*

(i) $K : (\text{LD})_M$ *and* (ii) *for any* $A \subset M$, $A : (\text{LD})_M \iff A \cup K : (\text{LD})_M$.

Proof. (1) The statement (i) is obvious.

(ii) For any $X \subset A$, it follows that $X \subset_u N$ in M and that $\text{LD}(X, N) \iff \text{LD}(X, M)$ by Lemma 3.2. This means the conclusion.

(2) (i) Take any $X \subset A \cup B$ and any admissible tuple (X, W', W, Z, Y, M). We have to find a neighborhood $\mathcal{W}$ of i_W in $\mathcal{E}_\#^u(W, M; Y)$ and an admissible deformation $\varphi : \mathcal{W} \times [0, 1] \longrightarrow \mathcal{E}_\#^u(W, M; Z)$ of $\mathcal{W}$ on X. There exists $\varepsilon > 0$ such that $O_\varepsilon(A) \subset_u U$, $O_\varepsilon(X) \subset_u W'$ and $O_\varepsilon(Z) \subset_u Y$.

(a) Let $X_1 = O_\varepsilon(X \cap A)$ and $Z_1 = O_\varepsilon(Z)$. Then, (X_1, W', W, Z_1, Y, M) is an admissible tuple and $X_1 \subset U$. Since $U : (\text{LD})_M$, there exists a neighborhoodd $\mathcal{W}_1$ of i_W in $\mathcal{E}_\#^u(W, M; Y)$ and an admissible deformation

$$\psi : \mathcal{W}_1 \times [0, 1] \longrightarrow \mathcal{E}_\#^u(W, M; Z_1)$$

of $\mathcal{W}_1$ on X_1.

(b) Let $X_2 = X \cap B$, $Z_2 = Z \cup (X \cap A)$ and $Y_2 = Z_1 \cup X_1$. Then $O_\varepsilon(Z_2) = O_\varepsilon(Z) \cup O_\varepsilon(X \cap A) = Z_1 \cup X_1 = Y_2$, so that $(X_2, W', W, Z_2, Y_2, M)$ is an admissible tuple and $X_2 \subset B$. Since $B : (\text{LD})_M$, there exists a neighborhood $\mathcal{W}_2$ of i_W in $\mathcal{E}_\#^u(W, M; Y_2)$ and an admissible deformation

$$\chi : \mathcal{W}_2 \times [0, 1] \longrightarrow \mathcal{E}_\#^u(W, M; Z_2)$$

of $\mathcal{W}_2$ on X_2.

Since $\psi_1(h) = \text{id}$ on $W \cap Y_2 = (W \cap Z_1) \cup X_1$ for any $h \in \mathcal{W}_1$, we have the map $\psi_1 : \mathcal{W}_1 \to \mathcal{E}_\#^u(W, M; Y_2)$. Since $\psi_1(i_W) = i_W \in \mathcal{W}_2$, we can find a neighborhood $\mathcal{W}$ of i_W in $\mathcal{W}_1$ such that $\psi_1(\mathcal{W}) \subset \mathcal{W}_2$. Finally, the required admissible deformation of $\mathcal{W}$ on X is defined by

$$\varphi : \mathcal{W} \times [0, 1] \longrightarrow \mathcal{E}_\#^u(W, M; Z) : \quad \varphi_t(h) = \begin{cases} \psi_{2t}(h) & (t \in [0, 1/2]) \\ \chi_{2t-1}(\psi_1(h)) & (t \in [1/2, 1]). \end{cases}$$

Note that $\varphi_1(h) = \chi_1(\psi_1(h)) = \text{id}$ on $(W \cap Z_2) \cup X_2 \supset (X \cap A) \cup (X \cap B) = X$.

The statement (ii) follows from (i).

(3) (i) By the Edwards - Kirby's local deformation theorem [4, Theorem 5.1] and [8, Remark 2.1, Complement to Theorem 2.1] the condition $LD(X, M)$ holds for any relatively compact subset X of M. Since K is relatively compact, so is any subset X of K and $LD(X, M)$ holds.

(ii) Take any compact neighborhood L of $cl_M K$ in M. Then $K \subset_u L$ and $L : (\mathrm{LD})_M$ by (1). Thus, the conclusion follows from (1)(i) and (2)(i).   □

For submanifolds of M we have the following conclusions.

Corollary 3.1.

(1) *Suppose $M = A \cup B$, A, B are topological n-manifolds possibly with boundary and $A - B \subset_u A$. Then, A, $B : (\mathrm{LD}) \implies M : (\mathrm{LD})$.*

(2) *Suppose $L \subset M$, L is a topological n-manifolds possibly with boundary and $M - L$ is relatively compact in M. Then*
   (i) $L : (\mathrm{LD}) \implies M : (\mathrm{LD})$ *and* (ii) *if L is closed in M, then $M : (\mathrm{LD}) \implies L : (\mathrm{LD})$*

(3) *Suppose $M = K \cup \cup_{i=1}^m L_i$, K is compact, each L_i is an n-manifold possibly with boundary and $d(L_i, L_j) > 0$ for any $i \neq j$. Then,*
   (i) $L_i : (\mathrm{LD})$ $(i = 1, \cdots, m) \implies M : (\mathrm{LD})$ *and*
   (ii) *if each L_i is closed in M, then $M : (\mathrm{LD}) \implies L_i : (\mathrm{LD})$ $(i = 1, \cdots, m)$.*

Proof. (1) Let $A_1 = A - B$. There exists $\varepsilon > 0$ such that $O_{3\varepsilon}(A_1) \subset A$. Since $A : (\mathrm{LD})_A$ and $O_{2\varepsilon}(A_1) \subset_u A$, it follows that $O_{2\varepsilon}(A_1) : (\mathrm{LD})_A$ and hence $O_{2\varepsilon}(A_1) : (\mathrm{LD})_M$. Let $B_1 = B - O_\varepsilon(A_1)$. Since $O_\varepsilon(B_1) \subset B$ and $B : (\mathrm{LD})_B$, it follows that $B_1 : (\mathrm{LD})_B$ and hence $B_1 : (\mathrm{LD})_M$ Therefore, by Proposition 3.1 (2) $M = O_\varepsilon(A_1) \cup B_1 : (\mathrm{LD})_M$.

(2) Since $K = cl_M(M - L)$ is compact, there exists a compact neighborhood C of K in M and $\varepsilon > 0$ such that $O_\varepsilon(K) \subset C$. Let $L_1 = L - C$. Since $O_\varepsilon(L_1) \subset L$, C is compact and $M = L_1 \cup C$, from Proposition 3.1 (1) and (3)(ii) it follows that

$$L : (\mathrm{LD})_L \implies L_1 : (\mathrm{LD})_L \iff L_1 : (\mathrm{LD})_M \iff M : (\mathrm{LD})_M.$$

When L is closed in M, since $C \cap L$ is compact and $L = L_1 \cup (C \cap L)$, it follows that

$$L : (\mathrm{LD})_L \iff L_1 : (\mathrm{LD})_L.$$

These implications exhibit the conclusions.

(3) Let $L = \cup_{i=1}^m L_i$ and take $\varepsilon > 0$ such that $d(L_i, L_j) > \varepsilon$ for any $i \neq j$. Then, $O_\varepsilon(L_i; L) = L_i$ for each $i = 1, \cdots, m$ and L is a topological n-manifold possibly with boundary. By Proposition 3.1 (1)(ii) for each i we have $L_i : (\mathrm{LD})_{L_i} \iff L_i : (\mathrm{LD})_L$. Hence, by Proposition 3.1 (1)(i) and (2)(ii)

$$L_i : (\mathrm{LD})_{L_i} \ (i = 1, \cdots, m) \iff L : (\mathrm{LD})_L.$$

Since $M = K \cup L$ and K is compact, from (2) it follows

(i) $L : (\mathrm{LD}) \implies M : (\mathrm{LD})$ and (ii) if L is closed in M, then $M : (\mathrm{LD})_M \implies L : (\mathrm{LD})_L$.

The assertions follow from these observations.   □

4. Examples

In this section we discuss some examples of manifolds with the property (LD).

4.1. Manifolds with geometric group actions.

In this section we show that a topological n-manifold possibly with boundary with a fixed metric has the local deformation property for uniform embeddings if it admits a locally geometric group action (Theorem 4.1). We refer to [1, Chapter I.8] for basic facts on geometric group actions. First we recall some related notions.

Throughout this section, $X = (X, d)$ is a locally compact separable metric space, G is a (discrete) group and $\Phi : G \times X \to X$ is a continuous action of G on X. As usual, for $g \in G$ and $x \in X$ the element $\Phi(g, x) \in X$ is denoted by gx. For a point $x \in X$ the orbit Gx and the isotropy subgroup G_x of x are defined by $Gx = \{gx \mid g \in G\}$ and $G_x = \{g \in G \mid gx = x\}$ respectively. More generally, for any subsets $H \subset G$ and $C \subset X$ let $HC = \{gx \mid g \in H, x \in C\}$ and $G_C = \{g \in G \mid gC = C\}$. Then, G_C is a subgroup of G and for any coset $\overline{g} \in G/G_C$ the subset $gC \subset X$ is well-defined.

The action Φ of G on X is called geometric if it is proper, cocompact and isometric. Here, Φ is (a) proper if $\{g \in G \mid gF \cap F \neq \emptyset\}$ is a finite set for any compact subset F of X, (b) cocompact if the quotient space X/G is compact, and (c) isometric if each $g \in G$ acts on X as an isometry. Note that (i) the action Φ is cocompact if and only if $M = GK$ for some compact subset K of M and (ii) if Φ is proper, then (α) G is a countable group since X is σ-compact, (β) for any nonempty compact subset C of X the family $\{gC \mid g \in G\}$ is locally finite and G_C is a finite subgroup of G, so (γ) the orbit Gx is discrete in X for any point $x \in X$.

In this article we work in a slightly more general setting.

Definition 4.1. We say that the action Φ of G on X is

(1) locally isometric if for every $x \in X$ there exists $\varepsilon > 0$ such that

$(\natural)_x$ each $g \in G$ maps $O_\varepsilon(x)$ isometrically onto $O_\varepsilon(gx)$, and

(2) locally geometric if it is proper, cocompact and locally isometric.

Remark 4.1. For $x \in X$, let $r_x = \sup\{\varepsilon \in [0, \infty] \mid \varepsilon : (\natural)_x\} \in [0, \infty]$. Then

(i) r_x itself satisfies the condtion $(\natural)_x$, so that $\{\varepsilon \in [0, \infty] \mid \varepsilon : (\natural)_x\} = [0, r_x]$ and

(ii) the action Φ is locally isometric if and only if $r_x > 0$ for each $x \in X$,

(iii) if $\gamma \in [0, r_x]$, then $O_\gamma(gx) = gO_\gamma(x)$ for each $g \in G$ and $O_\gamma(Fx) = FO_\gamma(x)$ for any subset $F \subset G$.

Lemma 4.1. *Suppose the action Φ is locally geometric and let $x \in X$ be any point.*

(1) *The orbit Gx is uniformly discrete. Hence, there exists $\varepsilon \in (0, r_x)$ such that the orbit Gx is 4ε-discrete.*

Let Λ be a complete set of representatives of cosets in G/G_x. For any ε as in (1) the following holds.

(2) (i) *$O_\varepsilon(Gx)$ is the disjoint union of open subsets $O_\varepsilon(gx) = gO_\varepsilon(x)$ ($g \in \Lambda$) and the family $\{O_\varepsilon(gx) \mid g \in \Lambda\}$ is 2ε-discrete.*

(ii) *The map $\pi : O_\varepsilon(Gx) \to O_\varepsilon(x)$ defined by*

$$\pi|_{O_\varepsilon(gx)} = g^{-1} : \ O_\varepsilon(gx) \cong O_\varepsilon(x) \quad \text{for each } g \in \Lambda$$

is a trivial metric covering projection.

In addition, when X is a topological n-manifold possibly with boundary, the following holds.

(3) *Let D be a closed n-disk neighborhood of x in $O_\varepsilon(x)$ and $\delta \in (0, \varepsilon/2)$ be such that $O_{2\delta}(x) \subset D$. Then $O_\delta(Gx) = GO_\delta(x)$ and it satisfies the condition* $(LD)_X$.

Proof. (1) Since Φ is proper and locally isometric, it follows that $r_x > 0$ and the orbit Gx is discrete. Hence, there exists $\delta \in (0, r_x)$ such that $O_\delta(x) \cap Gx = \{x\}$. Then Gx is δ-discrete. In fact, if $g, h \in G$ and $gx \neq hx$, then $g^{-1}hx \neq x$ so that $g^{-1}hx \notin O_\delta(x)$. Since $\delta \in (0, r_x)$, it follows that $gO_\delta(x) = O_\delta(gx)$ and so $hx \notin O_\delta(gx)$.

(2) (i) For any distinct $g, h \in \Lambda$ it follows that $gx \neq hx$, so $d(gx, hx) \geq 4\varepsilon$ since Gx is 4ε-discrete, hence $d(O_\varepsilon(gx), O_\varepsilon(hx)) \geq 2\varepsilon$.

(ii) Since $\varepsilon \in (0, r_x)$, each $g \in \Lambda$ induces an isometry $g : O_\varepsilon(x) \cong O_\varepsilon(gx)$. Hence, the restriction $\pi|_{O_\varepsilon(gx)} = g^{-1} : O_\varepsilon(gx) \cong O_\varepsilon(x)$ is a well-defined isometry. For each $y \in O_\varepsilon(x)$ the fiber of y is given by $\pi^{-1}(y) = \{gy \mid g \in \Lambda\}$. Since $\{O_\varepsilon(gx) \mid g \in \Lambda\}$ is 2ε-discrete and $gy \in O_\varepsilon(gx)$ for each $g \in \Lambda$, it follows that $\pi^{-1}(y)$ is also 2ε-discrete.

For any $y, y' \in O_\varepsilon(Gx)$ we have $d(\pi(y), \pi(y')) \leq d(y, y')$. In fact, if $y, y' \in O_\varepsilon(gx)$ for some $g \in \Lambda$, then $d(\pi(y), \pi(y')) = d(y, y')$ since $\pi|_{O_\varepsilon(gx)}$ is an isometry, and if $y \in O_\varepsilon(gx)$ and $y' \in O_\varepsilon(hx)$ for some distinct $g, h \in \Lambda$, then $d(\pi(y), \pi(y')) \leq \text{diam} \, O_\varepsilon(x) \leq 2\varepsilon \leq d(O_\varepsilon(gx), O_\varepsilon(hx)) \leq d(y, y')$.

(3) Since $\delta \in (0, r_x)$, we have $O_\delta(Gx) = GO_\delta(x)$. Let $N = \pi^{-1}(D)$. Since the restriction $\pi : N \to D$ is also a metric covering projection, it follows that N is a topological n-manifold possibly with boundary and satisfies the condition (LD) by Theorem 2.1. Since $O_{2\delta}(x) \subset D$ and $2\delta \in (0, r_x)$, we have

$$O_{2\delta}(Gx) = O_{2\delta}(\Lambda x) = \Lambda O_{2\delta}(x) \subset \Lambda D = N.$$

Then, by Proposition 3.1 (1) (i),(ii) $O_\delta(Gx)$ satisfies the condition $(LD)_N$ and hence $(LD)_X$. $\qquad \square$

The following is the main result of this section.

Theorem 4.1. *Suppose that (M, d) is a (separable) topological n-manifold possibly with boundary with a metric d and that it admits a locally geometric group action. Then (M, d) has the local deformation property for uniform embeddings, that is, the following holds: if X is a subset of M, $W' \subset W$ are uniform neighborhoods of X in M and Z, Y are subsets of M such that Y is a uniform neighborhood of Z, then there exists a neighborhood $\mathcal{W}$ of the inclusion map $i_W : W \subset M$ in $\mathcal{E}_{\#}^u(W, M; Y)$ and a homotopy $\varphi : \mathcal{W} \times [0, 1] \longrightarrow \mathcal{E}_{\#}^u(W, M; Z)$ such that*

(1) *for each $h \in \mathcal{W}$* (i) $\varphi_0(h) = h$, (ii) $\varphi_1(h) = \text{id}$ *on* X,

(iii) $\varphi_t(h) = h$ *on* $W - W'$ *and* $\varphi_t(h)(W) = h(W)$ $(t \in [0, 1])$,

(iv) *if $h = \text{id}$ on $W \cap \partial M$, then $\varphi_t(h) = \text{id}$ on $W \cap \partial M$ $(t \in [0, 1])$,*

(2) $\varphi_t(i_W) = i_W \quad (t \in [0,1])$.

Proof. By Lemma 4.1 (3) for each point $x \in M$ there exists $\varepsilon(x) > 0$ such that $O_{\varepsilon(x)}(Gx) = GO_{\varepsilon(x)}(x)$ and $O_{2\varepsilon(x)}(Gx) : (LD)_M$. Since the group action is cocompact, $M = GK$ for some compact subset $K \subset M$ and there exist finitely many points $x_1, \cdots, x_m \in K$ such that $K \subset \cup_{i=1}^m O_{\varepsilon(x_i)}(x_i)$. Then $M = GK = \cup_{i=1}^m O_{\varepsilon(x_i)}(Gx_i)$ and, since $O_{2\varepsilon(x_i)}(Gx_i) : (LD)_M$ for $i = 1, \cdots, m$, Proposition 3.1 (2) (ii) implies that $M : (LD)$. This completes the proof. $\square$

4.2. **Some examples.**

Example 4.1. (The half space $\mathbb{R}_{\geq 0}^n$)

(1) The Euclidean space $\mathbb{R}^n$ with the standard Euclidean metric admits the canonical geometric action of $\mathbb{Z}^n$ (and the associated Riemannian covering projection $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ onto the flat torus). Therefore, $\mathbb{R}^n$ has the property (LD). By Corollary 3.1 (2) the Euclidean ends $\mathbb{R}_r^n = \mathbb{R}^n - O_r(\mathbf{0})$ $(r > 0)$ also have the property (LD).

(2) The half space $\mathbb{R}_{\geq 0}^n = \{\mathbf{x} \in \mathbb{R}^n \mid x_n \geq 0\}$ has the property (LD). In fact, consider the subspaces $A = \mathbb{R}^{n-1} \times [0,1]$ and $B = \mathbb{R}^{n-1} \times [1, \infty)$. Since $\mathbb{R}^n : (LD)_{\mathbb{R}^n}$ and $B \subset_u \mathbb{R}_{\geq 0}^n$ in $\mathbb{R}^n$, it follows that $B : (LD)_{\mathbb{R}^n}$ and so $B : (LD)_{\mathbb{R}_{\geq 0}^n}$. On the other hand, since the boundary collar $E = \mathbb{R}^{n-1} \times [0,3] \subset \mathbb{R}^n$ admits the canonical geometric action of $\mathbb{Z}^{n-1}$ defined by

$$m \cdot (y, z) = (y + m, z) \quad (m \in \mathbb{Z}^{n-1}, (y, z) \in E),$$

E has the property $(LD)_E$. Then $F = \mathbb{R}^{n-1} \times [0,2]$ also has the properties $(LD)_E$ and $(LD)_{\mathbb{R}_{\geq 0}^n}$, since $F \subset_u E$ in $\mathbb{R}_{\geq 0}^n$. Therefore, by Proposition 3.1(2)(i) the half space $\mathbb{R}_{\geq 0}^n = A \cup B$ has the property (LD) in itself.

Example 4.2. (Ends of manifolds)

(1) Suppose (M, d) is a connected topological n-manifold possibly with boundary with a metric d and N is a compact n-submanifold of M. We assume that $\mathrm{Fr}_M N$ is locally flat and transversal to ∂M so that $\mathrm{Fr}_M N$ is a proper $(n-1)$-submanifold of M and $L = cl_M(M - N)$ is also an n-submanifold of M. Then, L has only finitely many connected components $K_1, \cdots, K_k, L_1, \cdots, L_l$, where K_i's are compact and L_j's are non-compact. The enlargement $N_1 = N \cup \left(\cup_{i=1}^k K_i \right)$ is also a compact n-submanifold of M (which is connected if N is connected) and $M = N_1 \cup \left(\cup_{j=1}^l L_j \right)$. From Corollary 3.1 (3) it follows that, whenever $d(L_i, L_j) > 0$ for any $i \neq j$,

$$M : (LD) \iff L_j : (LD) \ (j = 1, \cdots, l).$$

(2) Suppose N is a compact topological n-manifold with nonempty boundary and C_i $(i = 1, \cdots, m)$ is a collection of connected components of the boundary ∂N. Consider the non-compact n-manifold M obtained from N by attaching an open collar $L_i = C_i \times [0, \infty)$ along C_i for each $i = 1, \cdots, m$. The ends of M are in 1-1 correspondence with the collars L_i $(i = 1, \cdots, m)$. Suppose M is equipped with a metric d such that $d(L_i, L_j) > 0$ for any $i \neq j$. Then $M : (LD)$ if and only if each $L_i : (LD)$.

Example 4.3. (Cylindrical ends)

Suppose (N, ρ) is a compact topological n-manifold possibly with boundary with a metric ρ.

(1) The product $M = N \times \mathbb{R}$ has the natural metric d defined by

$$d((x, t), (y, s)) = \sqrt{\rho(x, y)^2 + |t - s|^2}.$$

The group $\mathbb{Z}$ acts on M geometrically by

$$m \cdot (x, t) = (x, t + m) \quad ((x, t) \in M, m \in \mathbb{Z}).$$

Therefore, M has the property (LD).

(2) The half product $N \times [0, \infty)$ with the metric d defined in (1) is called a cylindrical end (or a product end). Any cylindrical end has the property (LD) by (1) and Example 4.2 (1). Therefore, in Example 4.2 (2), if each (L_i, d) is a cylindrical end, then M has the property (LD).

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GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY, KYOTO INSTITUTE OF TECHNOLOGY, KYOTO, 606-8585, JAPAN

E-mail address: yagasaki@kit.ac.jp